Details on Liouville Transformation

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The one dimensional wave equation modeling a string of variable mass density is

$$\rho(x)u_{tt}(x,t) = u_{xx}(x,t)$$

where $\rho(x) > 0$ for $x \in [0,1]$. Using Dirichlet boundary conditions, $u(0,t) = u(1,t) = 0$, and applying separation of variables, the differential equation becomes the eigenvalue problem

$$-y''(x) = \lambda\rho(x)y(x)\ .$$

The Liouville transformation to change this equation into the form

$$-w'' + q(x)w = \lambda w$$

requires $\rho \in \mathcal{C}^2$[0,1]. If that condition is satisfied, the transformation uses the change of dependent and independent variables given by

$$t = \int_0^x \sqrt{\rho(s)}ds \ \text{ and } \ w(t) = y(x)[\rho(x)]^{1/4}\ .$$

Taking the derivative once,

$$\frac{dw}{dt}\frac{dt}{dx} = \frac{1}{4}y(x)[\rho(x)]^{-3/4}\frac{d\rho}{dx} + \frac{dy}{dx}[\rho(x)]^{1/4}$$

(suppressing the argument of x for now)

$$\frac{dw}{dt}\rho^{1/2} = \frac{1}{4}y\rho^{-3/4}\rho' + y'\rho^{1/4}$$

Taking the derivative again,

$$\frac{d^2w}{dt^2}\rho + \frac{1}{2}\frac{dw}{dt}\rho^{-1/2}\rho' = \frac{1}{2}y'\rho^{-3/4}\rho' - \frac{3}{16}y\rho^{-7/4}(\rho')^2 + \frac{1}{4}y\rho^{-3/4}\rho'' + y''\rho^{1/4}$$

Simplifying terms,

$$\frac{d^2w}{dt^2}\rho + \frac{\rho'}{2\rho^{1/2}}\left(\frac{y\rho'}{4\rho^{5/4}} + \frac{y'}{\rho^{1/4}}\right) = \frac{1}{2}y'\rho^{-3/4}\rho' - \frac{3}{16}y\rho^{-7/4}(\rho')^2 + \frac{1}{4}y\rho^{-3/4}\rho'' + y''\rho^{1/4}$$

$$\frac{d^2w}{dt^2}\rho + \frac{\rho'}{2\rho^{1/2}}\left(\frac{y\rho'}{4\rho^{5/4}} + \frac{y'}{\rho^{1/4}}\right) = \frac{1}{2}y'\rho^{-3/4}\rho' - \frac{3}{16}y\rho^{-7/4}(\rho')^2 + \frac{1}{4}y\rho^{-3/4}\rho'' - \lambda y\rho^{5/4}$$

$$\frac{d^2w}{dt^2}\rho = \frac{-5}{16}y\rho^{-7/4}(\rho')^2 + \frac{1}{4}y\rho^{-3/4}\rho'' - \lambda y\rho^{5/4}$$

$$\frac{d^2w}{dt^2} = \frac{-5}{16}y\rho^{-11/4}(\rho')^2 + \frac{1}{4}y\rho^{-7/4}\rho'' - \lambda y\rho^{1/4}$$

$$\frac{d^2w}{dt^2} = \frac{-5}{16}y\rho^{1/4}\rho^{-3}(\rho')^2 + \frac{1}{4}y\rho^{1/4}\rho^{-2}\rho'' - \lambda y\rho^{1/4}$$

$$\frac{d^2w}{dt^2} = \frac{-5}{16}w\rho^{-3}(\rho')^2 + \frac{1}{4}w\rho^{-2}\rho'' - \lambda w$$

Thus

$$-\frac{d^2w}{dt^2} + \left(\frac{1}{4}\rho^{-2}\rho'' - \frac{5}{16}\rho^{-3}(\rho')^2\right) w = \lambda w$$

This is in the form of the Sturm-Liouville equation,

$$-\frac{d^2w}{dt^2} + q(t)w = \lambda w$$

with

$$q(t) = \frac{\rho''(x)}{4[\rho(x)]^2} - \frac{5[\rho'(x)]^2}{16[\rho(x)]^3} \; .$$

Reference

- E. Hille, Lectures on Ordinary Differential Equations, Addison-Wesley Publishing Co., Reading, 1969, p 340.